B.Sc. Part I Paper I

TRIGONOMETRY

Summation of Series (contd.)

Q. If a, b, c are the sides of a triangle, show that the sum of the series

$$\log a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{1}{3}\frac{b^3}{a^3}\cos 3C$$

---- to ∞ is $\log c$.

Soln. Let the given series

$$= \log a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{1}{3}\frac{b^3}{a^3}\cos 3C - \text{-- to } \infty$$

$$= \log a - C \text{ (say).}$$

$$\therefore C = \frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{1}{3}\frac{b^3}{a^3}\cos 3C + \text{-- to } \infty$$

$$\text{Let } S = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{1}{3}\frac{b^3}{a^3}\sin 3C + \text{-- to } \infty$$

$$\Rightarrow C + iS = \frac{b}{a}(\cos C + i\sin C) + \frac{b^2}{2a^2}(\cos 2C + i\sin 2C)$$

$$+ \frac{1}{3}\frac{b^3}{a^3}(\cos 3C + i\sin 3C) + \text{-- to } \infty$$

$$\Rightarrow C + iS = \frac{b}{a}e^{iC} + \frac{1}{2}\frac{b^2}{a^2}e^{2iC} + \frac{1}{3}\frac{b^3}{a^3}e^{3iC} + \text{-- to } \infty$$

$$\text{Let } \frac{b}{a}e^{iC} = x$$

$$\Rightarrow C + iS = x + \frac{x^2}{2} + \frac{x^3}{3} + \text{-- to } \infty$$

$$\boxed{\begin{aligned}\log(1-x) &= -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots \text{ to } \infty\\ &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots \text{ to } \infty\right)\end{aligned}}$$

$$\therefore\ C+iS = -\log(1-x)$$

$$\Rightarrow C+iS = -\log\left[1-\frac{b}{a}e^{ic}\right]$$

$$= -\log\left[1-\frac{b(\cos c+i\sin c)}{a}\right]$$

$$= -\log\left[\frac{a-b\cos c-bi\sin c}{a}\right]$$

$$= \log a - \log(a-b\cos c - ib\sin c)$$

$$\Rightarrow C+iS = \log a - \log\{(a-b\cos c) - ib\sin c\}$$

$$\boxed{\log(\alpha - i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) - i\tan^{-1}\frac{\beta}{\alpha}}$$

$$\Rightarrow C+iS = \log a - \frac{1}{2}\log\{(a-b\cos c)^2+b^2\sin^2 c\} + i\tan^{-1}\frac{b\sin c}{a-b\cos c}$$

$$\Rightarrow C+iS = \log a - \frac{1}{2}\log\{a^2-2ab\cos c+b^2\} + i\tan^{-1}\frac{b\sin c}{a-b\cos c} \quad \text{——— (A)}$$

$$\boxed{\because \cos C = \frac{a^2+b^2-c^2}{2ab}}$$

$\Rightarrow a^2+b^2-c^2 = 2ab\cos C$

$\Rightarrow a^2 - 2ab\cos C + b^2 = c^2$

using this in eq(A), we get

$$C + iS = \log a - \frac{1}{2}\log c^2 + i\tan^{-1}\frac{b\sin C}{a-b\cos C}$$

$$\Rightarrow C + iS = \log a - \log c + i\tan^{-1}\frac{b\sin C}{a-b\cos C}$$

Equating real parts, we have

$$C = \log a - \log c$$

$\Rightarrow$ The given series $= \log a - C$

$= \log a - (\log a - \log c)$

$= \log c$